A fixed point's theorem on three complete metric spaces

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Abstract. A fixed point theorem for three mappings on three metric spaces is proved. This result is a modification of the result of Nešić′ [1] from two mappings of a metric space into itself, to three mappings of different metric spaces. We have modified the methods used by Nešić′ [1] and by Jain, Shrivastava and Fischer [3]. We also show that the Theorem of Nung [2] is a corollary of our result and that it is sufficient the continuity of only one of the mappings. An application of our result is presented.
Keywords: fixed point, metric space, complete metric space

1. Introduction

In [1], the following theorem is proved:

Theorem 1.1 *Let* (X,d) *be a metric space and* S,T *be two mappings of* X *into itself, satisfying the following inequality:*

$$[1+pd(x,y)]d(Sx,Ty) \le p[d(x,Sx)d(y,Ty)+d(x,Ty)d(y,Sx)]+$$
$$+q\max\{d(x,y),d(x,Sx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Sx)]\}$$

for all $x,y\in X$, *where* $p\ge 0$ *and* $0\le q<1$.

If (X,d) *is* (S,T)*-orbitally complete metric space, then* S *and* T *have an unique common fixed point* u *in* X.

In [2], the following theorem is proved:

Theorem 1.1 *Let* $(X,d_1),(Y,d_2),(Z,d_3)$ *be three complete metric spaces and* T *be a continuous mapping of* X *into* Y, S *a continuous mapping of* Y *into* Z *and* R *be a continuous of* Z *into* X, *satisfying the following inequality:*

$$d_1(RSTx,RSy) \le c\max\{d_1(x,RSy),d_1(x,RSTx),d_2(y,Tx),d_3(Sy,STx)\}$$
$$d_2(TRSy,TRz) \le c\max\{d_2(y,TRz),d_2(y,TRSy),d_3(Z,Sy),d_1(Rz,RSy)\}$$
$$d_3(STRz,STx) \le c\max\{d_3(z,STx),d_3(z,STRz),d_1(x,Rz),d_2(Tx,TRz)\}$$

for all $x\in X,y\in Y$ *and* $z\in Z$, *where* $0\le c<1$. *Then* RST *has an unique fixed point* $u\in X$, TRS *has an unique fixed point* $v\in Y$ *and* STR *has an unique fixed point* $w\in Z$. *Further,* $Tu=v,Sv=w$ *and* $Rw=u$.

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In this paper we will give a generalization of Theorem 1.2 modifying the results of Nešic′ [1]. We will also show that in Theorem 1.2 it is not nescessary the continuity of the three mappings, but it is sufficient the continuity of only one of them.

An application of our result is presented.

2. Main results

Theorem 2.1 *Let* $(X,d_1),(Y,d_2),(Z,d_3)$ *be three complete metric spaces and* $T:X\to Y, S:Y\to Z$ *and* $R:Z\to X$ *be three mappings from which at least one of them is continuous, satisfying the following inequality:*

$$[1 + pd_1(x,RSy)+pd_2(y,Tx)]d_1(RSy,RSTx)\le$$
$$\le p[d_1(x,RSy)d_3(Sy,STx)+d_1(x,RSTx)d_2(y,TRSy)+d_1(x,RSy)d_2(y,Tx)]+$$
$$+\ q\max\{d_1(x,RSy),d_1(x,RSTx),d_2(y,Tx),d_3(STx,Sy)\} \tag{1}$$

$$[1 + pd_2(y,TRz)+pd_3(z,Sy)]d_2(TRz,TRSy)\le$$
$$\le p[d_2(y,TRz)d_1(Rz,RSy)+d_2(y,TRSy)d_3(z,STRz)+d_2(y,TRz)d_3(z,Sy)]+$$
$$+\ q\max\{d_2(y,TRz),d_2(y,TRSy),d_3(z,Sy),d_1(RSy,Rz)\} \tag{2}$$

$$[1 + pd_3(z,STx)+pd_1(x,Rz)]d_3(STx,STRz)\le$$
$$\le p[d_3(z,STx)d_2(Tx,TRz)+d_3(z,STRz)d_1(x,RSTx)+d_3(z,STx)d_1(x,Rz)]+$$
$$+\ q\max\{d_3(z,STx),d_3(z,STRz),d_1(x,Rz),d_2(TRz,Tx)\} \tag{3}$$

for all $x\in X, y\in Y, z\in Z$, *where* $p\ge 0$ *and* $0\le q<1$. *Then* RST *has an unique fixed point* $\alpha\in X$, TRS *has an unique fixed point* $\beta\in Y$ *and* STR *has an unique fixed point* $\gamma\in Z$. *Further,* $T\alpha=\beta, S\beta=\gamma$ *and* $R\gamma=\alpha$.

Proof. Let $x_0\in X$ be an arbitrary point. We define the sequences $(x_n),(y_n)$ and (z_n) in X,Y and Z respectively as follows:

$$x_n=(RST)^n x_0,\ y_n=Tx_{n-1},\ z_n=Sy_n$$

for $n=1,2,...$

By the inequality (2), for $y=y_n$ and $z=z_{n-1}$ we get:

$$[1 + pd_2(y_n,y_n)+pd_3(z_{n-1},z_n)]d_2(y_n,y_{n+1})\le$$
$$\le p[d_2(y_n,y_n)d_1(x_{n-1},x_n)+d_2(y_n,y_{n+1})d_3(z_{n-1},z_n)+d_2(y_n,y_n)d_3(z_{n-1},z_n)]+$$
$$+\ q\max\{d_2(y_n,y_n),d_2(y_n,y_{n+1}),d_3(z_{n-1},z_n),d_1(x_n,x_{n-1})\}$$

from which it follows:

$$d_2(y_n,y_{n+1})\le q\max\{d_2(y_n,y_{n+1}),d_1(x_n,x_{n-1}),d_3(z_n,z_{n-1})\}=q\max A$$

where $A=\{d_2(y_n,y_{n+1}),d_1(x_n,x_{n-1}),d_3(z_n,z_{n-1})\}$.

If $\max A=d_2(y_n,y_{n+1})$, then we have:

$$d_2(y_n,y_{n+1})\le qd_2(y_n,y_{n+1})$$

and since $0\le q<1$, it follows $d_2(y_n,y_{n+1})=0$.

Thus we have:

$$d_2(y_n, y_{n+1}) \le q \max\{d_1(x_n, x_{n-1}), d_3(z_n, z_{n-1})\} \qquad (4)$$

In the same way, by (3), for $x = x_{n-1}$ and $z = z_n$, we get:

$$[1 + pd_3(z_n, z_n) + pd_1(x_{n-1}, x_n)]d_3(z_n, z_{n+1}) \le$$
$$\le p[d_3(z_n, z_n)d_2(y_n, y_{n+1}) + d_3(z_n, z_{n+1})d_1(x_{n-1}, x_n) + d_3(z_n, z_n)d_1(x_n, x_n)] +$$
$$+ q \max\{d_3(z_n, z_n), d_3(z_n, z_{n+1}), d_1(x_{n-1}, x_n), d_2(y_{n+1}, y_n)\}$$

from which we get:

$$d_3(z_n, z_{n+1}) \le q \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n)\} \qquad (5)$$

In the same way, by (1), for $y = y_n$ and $x = x_n$ we get:

$$[1 + pd_1(x_n, x_n) + pd_2(y_n, y_{n+1})]d_1(x_n, x_{n+1}) \le$$
$$\le p[d_1(x_n, x_n)d_3(z_n, z_{n+1}) + d_1(x_n, x_{n+1})d_2(y_n, y_{n+1}) + d_1(x_n, x_n)d_2(y_n, y_{n+1})] +$$
$$+ q \max\{d_1(x_n, x_n), d_1(x_n, x_{n+1}), d_2(y_n, y_{n+1}), d_3(z_{n+1}, z_n)\}$$

from which we get:

$$d_1(x_n, x_{n+1}) \le q \max\{d_1(x_n, x_{n+1}), d_2(y_n, y_{n+1}), d_3(z_n, z_{n+1})\}$$

and by (4) and (5) we have:

$$d_1(x_n, x_{n+1}) \le q \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n)\} \qquad (6)$$

Taking n equal with $n-1, n-2, \ldots.$, using (4), (5) and (6) we obtain:

$$d_1(x_n, x_{n+1}) \le q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}$$
$$d_2(y_n, y_{n+1}) \le q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}$$
$$d_3(z_n, z_{n+1}) \le q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}$$

Since $0 \le q < 1$, the sequences $(x_n), (y_n)$ and (z_n) are Cauchy sequences with limit α, β and γ in X, Y and Y respectively.

Suppose that the mapping S is continuous. Then by

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$$

we get:

$$S\beta = \gamma \qquad (7)$$

By (1), for $y = \beta$ and $x = x_n$ we get:

$$[1 + pd_1(x_n, RS\beta) + pd_2(\beta, y_{n+1})]d_1(RS\beta, x_{n+1}) \le$$
$$\le p[d_1(x_n, RS\beta)d_3(S\beta, z_{n+1}) + d_1(x_n, x_{n+1})d_2(\beta, TRS\beta) + d_1(x_n, RS\beta)d_2(\beta, y_{n+1})] +$$
$$+ q \max\{d_1(x_n, RS\beta), d_1(x_n, x_{n+1}), d_2(\beta, y_{n+1}), d_3(\gamma, S\beta)\}$$

Letting n tend to infinity, by the fact that $S\beta = \gamma$ we get:

$$[1+pd_1(\alpha,RS\beta)]d_1(\alpha,RS\beta)\le qd_1(\alpha,RS\beta)$$

$$d_1(\alpha,RS\beta)\le \frac{q}{1+pd_1(\alpha,RS\beta)}d_1(\alpha,RS\beta)$$

from which it follows:

$$d_1(\alpha,RS\beta)=0\Leftrightarrow RS\beta=\alpha \qquad (8)$$

since

$$\frac{q}{1+pd_1(\alpha,RS\beta)}\le q<1.$$

By (2), for $z=S\beta$ and $y=y_n$, we get:

$$[1+pd_2(y_n,TRS\beta)+pd_3(S\beta,z_n)]d_2(TRS\beta,y_{n+1})\le$$
$$\le p[d_2(y_n,TRS\beta)d_1(RS\beta,x_n)+d_2(y_n,y_{n+1})d_3(S\beta,STRS\beta)+d_2(y_n,TRS\beta)d_3(S\beta,z_n)]+$$
$$+\ q\max\{d_2(y_n,TRS\beta),d_2(y_n,y_{n+1}),d_3(S\beta,z_n),d_1(x_n,RS\beta)\}$$

Letting n tend to infinity, by (7) and (8) we get:

$$[1+pd_2(\beta,TRS\beta)]d_2(TRS\beta,\beta)\le qd_2(TRS\beta,\beta)$$

from which it follows $d_2(TRS\beta,\beta)=0$ or

$$TRS\beta=\beta \qquad (9)$$

By (7), (8), (9) we get:

$$TRS\beta = TR\gamma=T\alpha=\beta$$
$$STR\gamma = ST\alpha=S\beta=\gamma$$
$$RST\alpha = RS\beta=R\gamma=\alpha.$$

Thus, we proved that the points α,β,γ are fixed points of RST, TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

We emphasize the fact that it is sufficient the contiunity of only one of the mappings T,S and R.

Let we prove now the unicity of the fixed points α,β and γ.

Assume that there is α' a fixed point of RST different from α.

By (1), for $y=T\alpha$ and $x=\alpha'$, we get:

$$[1+pd_1(\alpha',RST\alpha)+pd_2(T\alpha,T\alpha')]d_1(RST\alpha,RST\alpha')\le$$
$$\le p[d_1(\alpha',RST\alpha)d_3(ST\alpha,ST\alpha')+d_1(\alpha',RST\alpha')d_2(T\alpha,TRST\alpha)+d_1(\alpha',RST\alpha)d_2(T\alpha,T\alpha')]+$$
$$+\ q\max\{d_1(\alpha',RST\alpha),d_1(\alpha',RST\alpha'),d_2(T\alpha,T\alpha'),d_3(ST\alpha',ST\alpha)\}$$

or

$$
\begin{aligned}
&[1 + pd_1(\alpha',\alpha)+pd_2(T\alpha,T\alpha')]d_1(\alpha,\alpha') \le \\
&\le p[d_1(\alpha',\alpha)d_3(ST\alpha,ST\alpha')+0+d_1(\alpha',\alpha)d_2(T\alpha,T\alpha')]+ \\
&\quad + q\max\{d_1(\alpha',\alpha),0,d_2(T\alpha,T\alpha'),d_3(ST\alpha',ST\alpha)\}
\end{aligned}
$$

or

$$
\begin{aligned}
&[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le \\
&\le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha')+ \\
&\quad + q\max\{d_1(\alpha,\alpha'),d_2(T\alpha,T\alpha'),d_3(ST\alpha,ST\alpha')\} \qquad (10)
\end{aligned}
$$

In respect of $\max\{d_1(\alpha,\alpha'),d_2(T\alpha,T\alpha'),d_3(ST\alpha,ST\alpha')\}=\max A$ we distinguish the following three cases:

Case 1. If $\max A = d_1(\alpha,\alpha')$, we have $d_3(ST\alpha,ST\alpha') \le d_1(\alpha,\alpha')$, and by (10) we obtain:

$$
\begin{aligned}
&[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le \\
&\le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha')+qd_1(\alpha,\alpha') \le \\
&\le pd_1(\alpha,\alpha')d_1(\alpha,\alpha')+qd_1(\alpha,\alpha').
\end{aligned}
$$

By the above we obtain $d_1(\alpha,\alpha') \le qd_1(\alpha,\alpha')$ and since $0 \le q < 1$ we get:

$$\alpha = \alpha' \qquad (11)$$

Case 2. If $\max A = d_2(T\alpha,T\alpha')$, we have $d_3(ST\alpha,ST\alpha') \le d_2(T\alpha,T\alpha')$, and by (10) we obtain:

$$
\begin{aligned}
&[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le \\
&\le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha')+qd_2(T\alpha,T\alpha') \le \\
&\le pd_1(\alpha,\alpha')d_2(T\alpha,T\alpha')+qd_2(T\alpha,T\alpha').
\end{aligned}
$$

or

$$[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le [q+pd_1(\alpha,\alpha')]d_2(T\alpha,T\alpha')$$

$$d_1(\alpha,\alpha') \le \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')}d_2(T\alpha,T\alpha').$$

from which it follows:

$$d_1(\alpha,\alpha') \le rd_2(T\alpha,T\alpha') \qquad (12)$$

where

$$0 \le r = \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')} < 1,$$

since $0 \le q < 1$.

Case 3. If $\max A = d_3(ST\alpha,St\alpha')$, then the inequality (10) takes the form:

$$[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha')+qd_3(ST\alpha,ST\alpha')$$

$$d_1(\alpha,\alpha') \le \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')}d_3(ST\alpha,ST\alpha')$$

$$d_1(\alpha,\alpha') \le rd_3(ST\alpha,ST\alpha') \qquad (13)$$

Continuing our argumentation for the Case 2, by (2) for $z = ST\alpha$ and $y = T\alpha'$ we have:

$$\begin{aligned}&[1 + pd_2(T\alpha',TRST\alpha)+pd_3(ST\alpha,ST\alpha')]d_2(TRST\alpha,TRST\alpha') \le \\ &\le p[d_2(T\alpha',TRST\alpha)d_1(RST\alpha,RST\alpha')+d_2(T\alpha',TRST\alpha')d_3(ST\alpha,STRST\alpha)+ \\ &\quad +d_2(T\alpha',TRST\alpha)d_3(ST\alpha,ST\alpha')]+q\max\{d_2(T\alpha',TRST\alpha), \\ &\quad d_2(T\alpha',TRST\alpha'),d_3(ST\alpha,ST\alpha'),d_1(RST\alpha',RST\alpha)\}\end{aligned}$$

or

$$\begin{aligned}&[1 + pd_2(T\alpha',T\alpha)+pd_3(ST\alpha,ST\alpha')]d_2(T\alpha,T\alpha') \le \\ &\le p[d_2(T\alpha',T\alpha)d_1(\alpha,\alpha')+d_2(T\alpha',T\alpha')d_3(ST\alpha,ST\alpha)+ \\ &\quad +d_2(T\alpha',T\alpha)d_3(ST\alpha,ST\alpha')]+q\max\{d_2(T\alpha',T\alpha), \\ &\quad d_2(T\alpha',T\alpha'),d_3(ST\alpha,ST\alpha'),d_1(\alpha',\alpha)\}\end{aligned}$$

or

$$\begin{aligned}&[1 + pd_2(T\alpha',T\alpha)]d_2(T\alpha,T\alpha')] \le pd_2(T\alpha',T\alpha)d_1(\alpha,\alpha')+ \\ &+q\max\{d_1(\alpha',\alpha),d_2(T\alpha',T\alpha),d_3(ST\alpha,ST\alpha')\}\end{aligned}$$

or

$$[1+pd_2(T\alpha',T\alpha)]d_2(T\alpha,T\alpha') \le pd_2(T\alpha',T\alpha)d_1(\alpha,\alpha')+q\max A \qquad (14)$$

In the Case 2, we have $\max A = d_2(T\alpha,T\alpha')$ and by (14) we obtain:

$$[1+pd_2(T\alpha',T\alpha)]d_2(T\alpha,T\alpha') \le pd_2(T\alpha',T\alpha)d_2(T\alpha',T\alpha)+qd_2(T\alpha,T\alpha')$$

or

$$d_2(T\alpha,T\alpha') \le qd_2(T\alpha,T\alpha').$$

Since $0 \le q < 1$, we obtain:

$$d_2(T\alpha,T\alpha') = 0$$

and by (13) it follows that $d_1(\alpha,\alpha') = 0$, so we obtain again the inequality (11).

In the Case 3, by (3) for $x = RST\alpha, z = ST\alpha'$ and in the same way we obtain:

$$[1+pd_3(ST\alpha',ST\alpha)]d_3(ST\alpha,ST\alpha') \le pd_3(ST\alpha',ST\alpha)d_2(T\alpha',T\alpha)+q\max A.$$

Since in this case $\max A = d_3(ST\alpha,ST\alpha')$, we have $d_2(T\alpha',T\alpha) \le d_3(ST\alpha,ST\alpha')$ and we obtain:

$$[1+pd_3(ST\alpha,ST\alpha')]d_3(ST\alpha,ST\alpha') \le pd_3(ST\alpha,ST\alpha')d_3(ST\alpha,ST\alpha')+qd_3(ST\alpha,ST\alpha')$$

from which it follows

$$d_3(ST\alpha,ST\alpha') \le qd_3(ST\alpha,ST\alpha').$$

Since $0 \le q < 1$ we take:

$$d_3(ST\alpha, ST\alpha') = 0$$

and by (13) it follows $d_1(\alpha, \alpha') = 0$. Thus, again, in this case the following equality holds:

$$\alpha = \alpha'.$$

In the same way, it is proved the unicity of β and γ.

Application. Let $X = Y = Z = [0,1] \subset R$ and the mappings defined as follows:

$$Tx = x, Rz = 1 \text{ and } Sy = \begin{cases} 1 & \text{for } y \in]0,1] \\ \frac{1}{2} & \text{for } y = 0 \end{cases}$$

We have:

$$RSy = 1, TRz = 1 \text{ and } STx = \begin{cases} 1 & \text{for } x \in]0,1] \\ \frac{1}{2} & \text{for } x = 0 \end{cases}$$

$$RSTx = 1, TRSy = 1 \text{ and } STRZ = 1.$$

We have to show now that T, R and S satisfy the conditions of Theorem 2.1 for $k = \frac{1}{2}$ and $q = \frac{3}{4}$. Indeed:

For every $x, y \in [0,1]$ we have $d_1(RSy, RSTx) = |1-1| = 0$. Then the verity of the inequality (1) is clear since its left side is 0.

The verity of the inequality (2) is clear too, since $d_2(TRz, TRSy) = 0, \forall x, y \in [0,1]$.

We consider now the inequality (3). We have

$$d_3(STx, STRz) = \begin{cases} |\frac{1}{2} - 1| = \frac{1}{2}, & \text{for } x = 0 \text{ and } 0 \le z \le 1 \\ |1-1| = 0, & \text{for } 0 < x \le 1 \text{ and } 0 \le z \le 1 \end{cases}$$

We distinguish two cases:

Case 1. For $x = 0$ and $0 \le z \le 1$, the inequality (3) takes the form

$$(1 + p|z - \frac{1}{2}| + p|0-1|)\frac{1}{2} \le p(|z - \frac{1}{2}| - |0-1| + |z-1| \cdot |0-1| + |z - \frac{1}{2}| \cdot |0-1|) +$$

$$+ q \max\{|z - \frac{1}{2}|, |z-1|, |0-1|, |1-0|\}.$$

We get

$$(1+p)\frac{1}{2} \le \frac{3p}{2}|z - \frac{1}{2}| + p|z-1| + q.$$

For $p=\frac{1}{2}$ and $q=\frac{3}{4}$, we obtain

$$\frac{3}{4}\leq\frac{3}{4}|z-\frac{1}{2}|+\frac{1}{2}|z-1|+\frac{3}{4}$$

or

$$0\leq\frac{3}{4}|z-\frac{1}{2}|+\frac{1}{2}|z-1|$$

for all $z\in[0,1]$.

Thus, the inequality (3) is satisfied.

Case 2. For $0<x\leq 1$ and $0\leq z\leq 1$, since $d_3(STx,STRz)=0$, the inequality (3) is satisfied.

Therefore, as a conlcusion, we have the mappings T,S and R satisfy all the conditions of the Theorem 2.1 for $p=\frac{1}{2}$ and $q=\frac{3}{4}$. The unique fixed point is 1 for each of the mappings RST,TRS and STR.

Corollary 2.2 *Theorem 1.2*[2] *is taken by Theorem 2.1 for* $p=0$. *Further, it is sufficient the continuity of only one of the three mappings.*

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